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LAPLACE TRANSFORM

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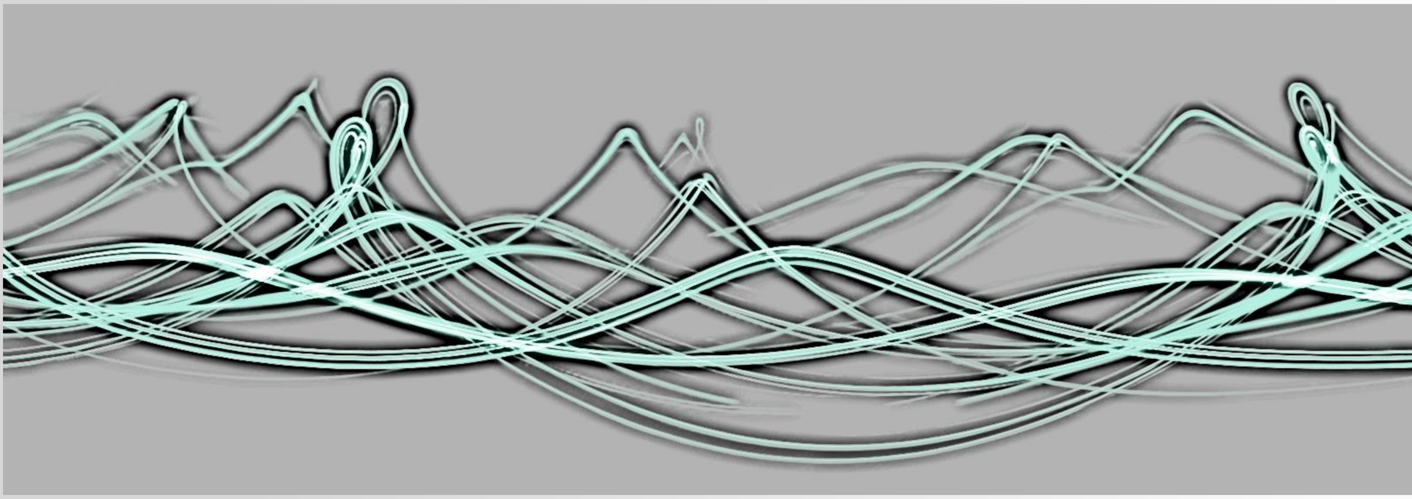
MATHEMATICS, SCIENCE & COMPUTER DEPARTMENT

Laplace Transform



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Preface

The Laplace Transform eBook is prepared by lectures in Mathematics, Science and Computer Department of Politeknik Sultan Salahuddin Abdul Aziz Shah. This eBook has been primarily written for polytechnic students who took engineering mathematics course in third semester. This eBook is very useful especially for electrical engineering students as their reference in other electrical courses for whom Laplace Transform continue to be very useful tool.

The book demands linear algebra and elementary knowledge of calculus which had been learned by polytechnics student in first and second semester. The content of this eBook is based on the syllabus prepared by Department of Polytechnic and Community College Education, Ministry of higher Education, Malaysia.

The eBook contains three main subtopics, apply definition of Laplace Transform, apply Laplace Transform and apply Inverse Laplace Transform. Each subtopic contains explanation in text and link of videos. Examples, solution, and exercises are also provided for better understanding during the revision.

We would like to dedicate special thanks to our Head Department for giving this opportunity and trust in produce the Laplace Transform eBook. Special thanks also dedicated to eLearning team, CRI units for guidance in producing eBook. It also a great pleasure to our Head of Mathematics Course as an expert in reviewing the content of the topic.

Department of Mathematic, Science and Computer
Politeknik Sultan Salahuddin Abdul Aziz Shah
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Table of Content

1.1 Laplace Transform

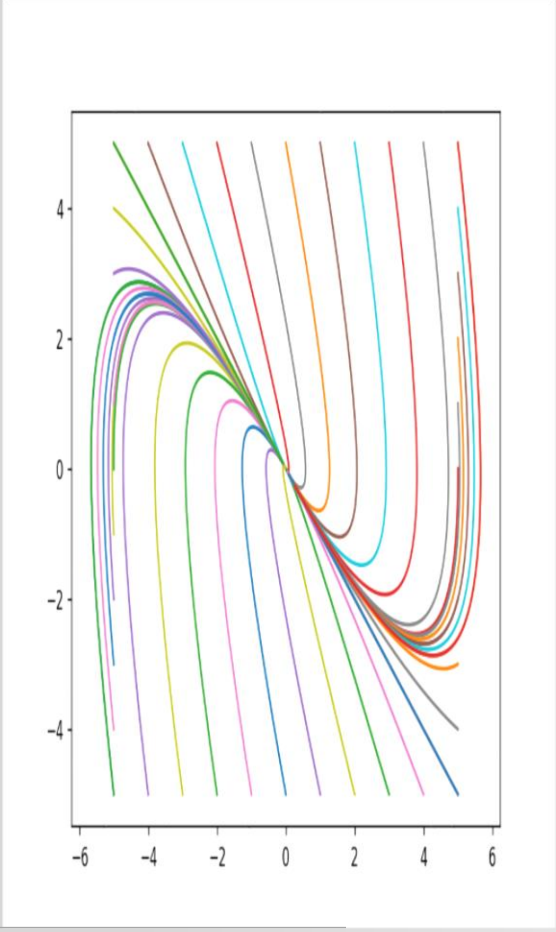
What is Laplace Transform?

The method of Laplace Transforms is a system that relies on algebra (rather than calculus based methods) to solve linear differential equations. It is a very powerful tool that enables us to readily deal with linear differential equations with discontinuous forcing functions.

Laplace transform is named in honour of the great French mathematician, Pierre Simon De Laplace (1749-1827)

Introduction

- The transform method is used to solve problems, which are difficult to solve directly.
- Therefore the Laplace method will transform the equation in order to solve it.
- Generally, Laplace transform is a method used to solve ODE. (Ordinary Differential Equations)

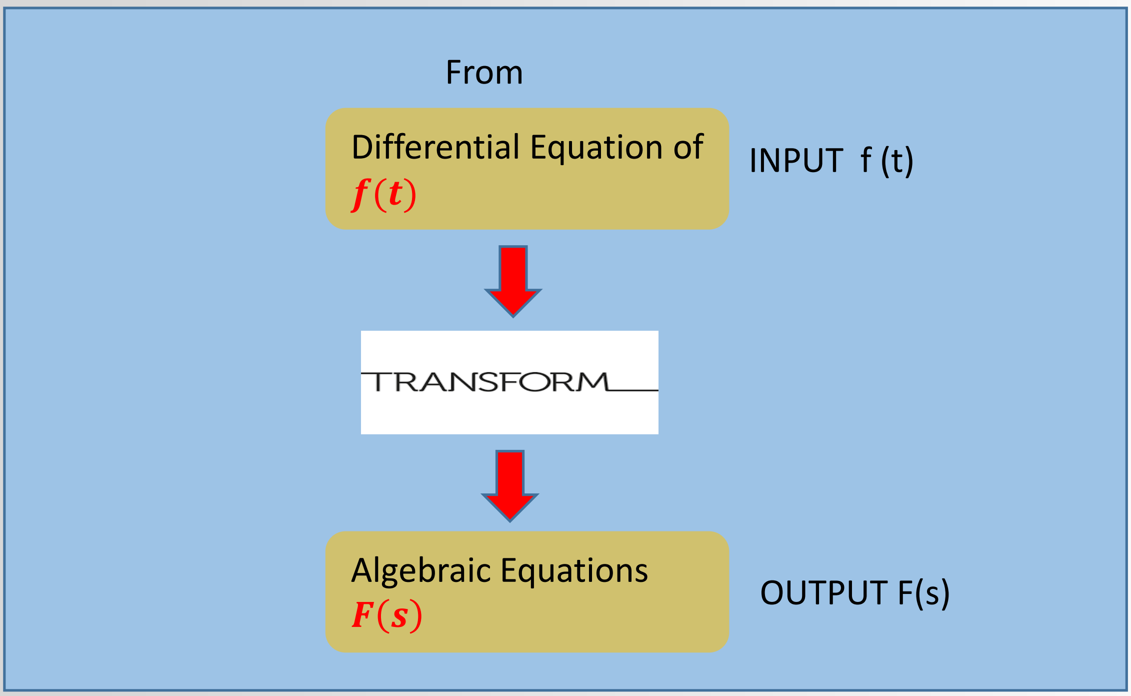


- The Laplace transform provides a useful method of solving certain types of differential equations when certain initial conditions are given, especially when the initial values are zero
- The Laplace transform is also very useful in the area of circuit analysis. It is often easier to analyse the circuit in its Laplace form.
- The techniques of Laplace transform are not only used in circuit analysis, but also in the following areas:

1. Proportional-Integral-Derivative (PID) controllers
2. DC motor speed control systems
3. DC motor position control systems
4. Second order systems of differential

How To Do Laplace Transform

- **Firstly,** the equation of time domain shape differential is changed to an algebraic equation of a frequency domain shape by taking Laplace transform of the equation. What we will learn in this section is to transform into algebraic equations.



- **Secondly**, After solving the algebra equation in the frequency domain, determine the Laplace transform of the unknown variable.

- **Finally**, convert this expression into time domain by taking inverse Laplace transform. We will learn this in the later sub-topic of Laplace transform,

What is the Definition of Laplace Transform?

- To solve the Laplace transform by definition means to solve a Laplace transform in the form of $L\{f(t)\}$ using the definition of the Laplace transform, where $\{F(s)\}$ is the Laplace transform of the function $f(t)$, s is a constant, and $f(t)$ is the given function.
- Therefore we need to apply the definition of the Laplace transform in order to transform any differential equation.
- Generally, Laplace transform is a method used to solve ODE. (Ordinary Differential Equations)

- The Laplace transform of $F = F(s)$ of a function $f = f(t)$ is defined by:

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt$$

- The integral is evaluated with respect to t, hence once the limits are substituted, what is left are in terms of s.

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt}$$

Laplace transform of f (t)

Equal to

Definition of Laplace transform

“

Definition of Laplace Transform

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt$$

”

How To Calculate Laplace Transform By Definition?

- To calculate Laplace Transform is applying the definition of Laplace transform

  $L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt$

- Substitute $f(t)$ which is stated in the question into the formula of Laplace transform definition.
- Integrate respectively
- The result is an algebraic equation, which is in simpler form and much easier to solve

Apply this Formula, substitute f (t) given in the question into the formula.

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt$$

$$\therefore L\{f(t)\} = F(s)$$

Integration Formulas

Formula:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$; $TRUE\ when\ n \neq -1$
2. $\int af(x)dx = a\int f(x)dx$
3. $\int a\ dx = ax + C$; $where\ a\ is\ constant$
4. $\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)\, dx$
5. $\int e^u dx = \frac{e^u}{\frac{du}{dx}} + C$

Examples and solution

Example 1

Calculate the Laplace Transform of $f(t) = e^{at}$ using the definition of the Laplace Transform.

Solution:

$$L\{e^{at}\} = F\ (s) = \int_0^{\infty} e^{-st}\ e^{at} dt$$

$$= \int_0^{\infty} e^{-st+at}\ dt$$

$$= \int_0^{\infty} e^{-(a-s)t}\ dt$$

$$= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{\infty} = \frac{1}{a-s}\left[e^{-(s-a)t}\right]_0^{\infty}$$

$$= \frac{1}{a-s}\left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right]$$

$$= \frac{1}{a-s}\left[\frac{1}{\infty} - 1\right]$$

$$= \frac{1}{a-s}\ (-1)$$

$$= \frac{1}{a-s}$$

Examples and solution

Example 2

Calculate the Laplace Transform of $f(t) = a$ using the definition of the Laplace Transform.
[*where a = constant*]

Solution:

$$L\{a\} = F\ (s) = \int_0^{\infty} e^{-st}\ a\ dt$$

$$= a \int_0^{\infty} e^{-st}\ dt$$

$$= a \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{-\ a}{s} \left[\frac{1}{e^{st}}\right]_0^{\infty}$$

$$= \frac{-a}{s} \left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right]$$

$$= \frac{-a}{s}\ (0 - 1)$$

$$= \frac{a}{s}$$

Examples and solution

Example 3

Calculate the Laplace Transform of $f(t) = m + 7e^{4t}$ using the definition of the Laplace Transform.

Solution:

$$= \int_0^{\infty} e^{-st}\,(m + 7e^{4t})dt$$

$$= \int_0^{\infty} (me^{-st} + 7e^{4t-st})\,dt$$

$$= m\left[\frac{e^{-st}}{-s}\right]_0^{\infty} + 7\left[\frac{e^{t(4-s)}}{4-s}\right]_0^{\infty}$$

$$= \frac{-m}{s}(e^{\infty} - e^{0}) + \frac{7}{4-s}\,(e^{\infty} - e^{0})$$

$$= \frac{-m}{s}(0 - 1) + \frac{7}{4-s}(0 - 1)$$

$$= \frac{m}{s} - \frac{7}{4-s}$$

$$= \frac{m}{s} + \frac{7}{s - 4}$$

Exercises and Answers

Question:

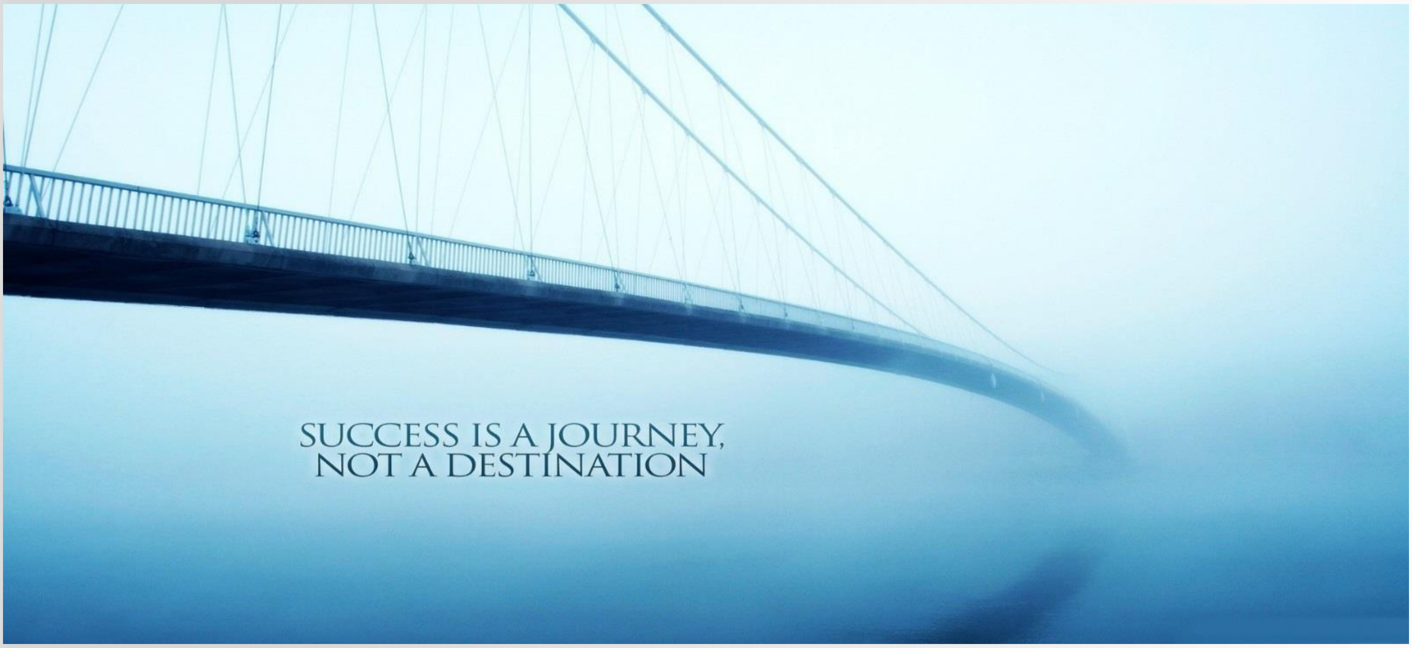
Calculate the Laplace Transform of $f(t)$ using the Definition of the Laplace Transform:

a) $f(t) = p$

b) $f(t) = -9$

c) $f(t) = e^{nt}$

d) $f(t) = 7e^{4t}$



Check Your Answers.

a) $F(s)=\frac{p}{s}$ b) $F(s) = \frac{-9}{s}$

c) $F(s) = \frac{1}{s-n}$ d) $F(s) = \frac{7}{s-4}$

1.2 Apply Laplace Transform

Solving Laplace transform by using Laplace transform table

- It is important to understand the use of table and the formula together.
- Each expression is derived by finding the infinite integral that we saw in the definition of Laplace Transform section before
- Another method that can be used in solving Laplace Transform is by using Laplace Transform table

LAPLACE TRANSFORM TABLE

No.	$f(t)$	$F(s)$	No.	$f(t)$	$F(s)$
1.	a	$\frac{a}{s}$	13.	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
2.	at	$\frac{a}{s^2}$	14.	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
3.	t^n	$\frac{n!}{s^{n+1}}$	15.	$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2}$
4.	e^{at}	$\frac{1}{s-a}$	16.	$\cosh\omega t$	$\frac{s}{s^2-\omega^2}$
5.	e^{-at}	$\frac{1}{s+a}$	17.	$e^{at}\sinh\omega t$	$\frac{\omega}{(s-a)^2-\omega^2}$
6.	te^{-at}	$\frac{1}{(s+a)^2}$	18.	$e^{-at}\sinh\omega t$	$\frac{\omega}{(s+a)^2-\omega^2}$
7.	$t^n \cdot e^{at}$, $n = 1,2,3..$	$\frac{n!}{(s-a)^{n+1}}$	19.	$e^{-at}\cosh\omega t$	$\frac{s+a}{(s+a)^2-\omega^2}$
8.	$t^n \cdot f(t)$	$(-1)^n \frac{d^n}{ds^n}[F(s)]$	20.	$f_1(t)+f_2(t)$	$F_1(s)+F_2(s)$
9.	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	21.	$\int_0^t f(u)du$	$\frac{F(s)}{s}$
10.	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	22.	$f(t-a)\,u(t-a)$	$e^{-as}F(s)$
11.	$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	23.	*First derivative*: $\frac{dy}{dt}, y'(t)$	$sY(s)-y(0)$
12.	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	24.	*Second derivative* $\frac{d^2y}{dt^2}, y''(t)$	$s^2Y(s)-sy(0)$ $-y'(0)$

Example 1

Use table to find the Laplace transform to the function $\mathcal{L}\left[\frac{1}{3}\right]$

Solution : $\mathcal{L}\left[\frac{1}{3}\right] = \mathcal{L}\{f(t)\}$

$$\text{Since } \frac{1}{3} \; is\ equal\ to\ \boldsymbol{a}\ (constant\ arbitrary)$$

$$f(t) = \frac{1}{3}$$

$$\mathcal{L}\left[\frac{1}{3}\right] = \frac{1/3}{s}$$

$$\mathcal{L}\left[\frac{1}{3}\right] = \frac{1}{3s}$$

Refer Laplace Transform Table

$f(t)$	$F(s)$
a	$\frac{a}{s}$

Example 2

Use table to find the Laplace transform to the function $\mathcal{L}[\cos 2t]$

Solution : $\mathcal{L}[\cos 2t] = \mathcal{L}\{f(t)\}$

$$\omega = 2$$

$$F(s) = \mathcal{L}[\cos 2t] = \frac{s}{s^2 + 2^2}$$

$$Therefore,$$

$$F(s) = \boldsymbol{\mathcal{L}[\cos 2t]} = \frac{\boldsymbol{s}}{\boldsymbol{s^2 + 4}}$$

Refer Laplace Transform Table

$f(t)$	$F(s)$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$

Example 3

Use table to find the Laplace transform to the function $\mathcal{L}[t^3]$

Solution : $\mathcal{L}[t^3] = \mathcal{L}[f(t)]$

$n = 3:$

$$F(s) = \mathcal{L}[t^3] = \frac{3!}{s^{3+1}}$$

$Therefore,$

$$F(s) = \boldsymbol{\mathcal{L}[t^3]} = \frac{\mathbf{6}}{\boldsymbol{s^4}}$$

Refer Laplace Transform Table

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
t^n	$\frac{n!}{s^{n+1}}$

Example 4

Use table to find the Laplace transform to the function $\mathcal{L}[3\cosh 2t]$

Solution : $\mathcal{L}[3\cosh 2t] = \mathcal{L}\{f(t)\}$

$\omega = 2$

$F(s) = \mathcal{L}[3\cosh 2t]$

$\mathcal{L}[3\cosh 2t] = 3\mathcal{L}[\cosh 2t]$

$Therefore,$

$$F(s) = 3\mathcal{L}[\cosh 2t] = \frac{3s}{s^2 - 2^2}$$

$$F(s) = \frac{3s}{s^2 - 4}$$

Refer Laplace Transform Table

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$

Theorem 1.2 (Linear Properties)

- The Laplace Transform has many interesting and useful properties, the most fundamental of which is linearity
- If $f_1(t)$ and $f_2(t)$ are two functions whose Laplace Transform exist, then

$$\boldsymbol{\mathcal{L}\{af_1(t) + bf_2(t)\} = a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_1(t)\}}$$

Where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary constant

Proof of Linearity properties:

$$\begin{aligned}
\mathcal{L}\{af_1(t) + bf_2(t)\} &= \int_0^\infty (af_1 + bf_2)\, e^{-st}\, dt \\
&= \int_0^\infty (af_1e^{-st} + bf_2e^{-st})\ dt \\
&= a\int_0^\infty af_1e^{-st}\, dt + b\int_0^\infty af_2e^{-st}\, dt \\
&= a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_1(t)\}
\end{aligned}$$

Linearity Properties

$$\boldsymbol{\mathcal{L}\{af_1(t) + bf_2(t)\} = a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_1(t)\}}$$

Example 1

Find the Laplace Transform of $f(t) = 3e^{2t} + 4\sin 5t$

Solution :

$$\mathcal{L}\{3e^{2t} + 4\sin 5t\}$$

$$= \mathcal{L}\{3e^{2t}\} + \mathcal{L}\{4\sin 5t\}$$

$$= 3\mathcal{L}\{e^{2t}\} + 4\mathcal{L}\{\sin 5t\}$$

$$= 3\frac{1}{s-2} + 4\frac{5}{s^2+5^2}$$

$$= \frac{3}{s-2} + \frac{20}{s^2+25}$$

Refer Laplace Transform Table

$f(t)$		$F(s)$
e^{2t}	e^{at}	$\frac{1}{s-a}$
$\sin 5t$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Example 2

Find the Laplace Transform of $f(t) = \sin t \cos t - 4$

Solution :

$$\mathcal{L}\{\sin t \cos t - 4\}$$

$$= \mathcal{L}\{\sin t \cos t\} - \mathcal{L}\{4\}$$

$$= \mathcal{L}\left\{\boxed{\frac{\sin 2t}{2}}\right\} + \mathcal{L}\{4\}$$

$$= \frac{1}{2}\mathcal{L}\{\sin 2t\} + \mathcal{L}\{4\}$$

$$= \left(\frac{1}{2}\right)\left(\frac{4}{s^2+16}\right) + \frac{4}{s}$$

$$= \frac{2}{s^2+16} - \frac{4}{s}$$

From Identity Trigonometry:

$$\sin 2x = 2\sin x \cos x$$

$$\frac{\sin 2x}{2} = \sin x \cos x$$

Refer Laplace Transform Table

$f(t)$		$F(s)$
4	a	$\frac{a}{s}$
$\sin 2t$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Example 3

Find the Laplace Transform of $f(t) = \cos 4t + \sin 7t$

Solution :

$$\mathcal{L}\{\cos 4t + \sin 7t\}$$

$$= \mathcal{L}\{\cos 4t\} + \mathcal{L}\{\sin 7t\}$$

$$= \frac{s}{s^2 + 4^2} + \frac{7}{s^2 + 7^2}$$

$$= \frac{s}{s^2 + 16} + \frac{7}{s^2 + 49}$$

Refer Laplace Transform Table

$f(t)$		$F(s)$
$\cos 4t$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin 7t$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$

Example 4

Find the Laplace Transform of $f(t) = t^3 - 4t^2 + 3t^5$

Solution :

$$\mathcal{L}\{t^3 - 4t^2 + 3t^5\}$$

$$= \mathcal{L}\{t^3\} - \mathcal{L}\{4t^2\} + \mathcal{L}\{3t^5\}$$

$$= \frac{3!}{s^4} - 4\left(\frac{2!}{s^3}\right) + 3\left(\frac{5!}{s^6}\right)$$

$$= \frac{6}{s^4} - \frac{8}{s^3} + \frac{120}{s^6}$$

Refer Laplace Transform Table

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$

Theorem 1.3 (First Shift Theorem)

- First Shift Theorem is the substitution $(s-a)$ for s in the transform corresponds to the multiplication of the original function to e^{at}
- If $\mathcal{L}\{f(t)\} = F(s)$ and a is constant, then
  $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
- If $\mathcal{L}\{f(t)\} = F(s)$ and a is constant, then
  $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$
- If $F(s-a)$ and $F(s+a)$ can be derived by substituting s with $(s-a)$ and $(s+a)$

First Shifting Properties

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \text{ where } s \to (s-a)$$

$$\mathcal{L}\{e^{-at} f(t)\} = F(s+a) \text{ where } s \to (s+a)$$

Proof of First Shifting properties:

$$\begin{aligned}
\mathcal{L}\{e^{at} f(t)\} &= \int_0^\infty e^{-st} \cdot e^{at} \cdot f(t)\, dt \\
&= \int_0^\infty e^{(-s+a)t} \cdot f(t)\, dt \\
&= \int_0^\infty e^{-(s-a)t} \cdot f(t)\, dt \\
&= \int_0^\infty e^{-pt} \cdot f(t)\, dt \\
&= F(p) \\
&= F(s-a)
\end{aligned}$$

Where $p = s - a$

Example 1

Find the Laplace Transform of $f(t) = \frac{4}{3}e^{2t} \cdot t^4$

Solution :

$$\mathcal{L}\left\{\frac{4}{3}e^{2t} \cdot t^4\right\}$$

Consider $f(t) = \frac{4}{3}t^4$

$$\mathcal{L}\left\{\frac{4}{3}t^4\right\} = \left(\frac{4}{3}\right)\frac{24}{s^5}$$

Refer Laplace Transform Table for t^n

$$= \frac{32}{s^5}$$

$$\therefore \mathcal{L}\left\{\frac{4}{3}e^{2t} \cdot t^4\right\} = \frac{32}{(s-2)^5}$$

$e^{2t}: a = 2$
$\therefore s \rightarrow (s-2)$

Example 2

Find the Laplace Transform of $f(t) = e^{-3t} \cdot \cosh 2t$

Solution :

$$\mathcal{L}\{e^{-3t} \cdot \cosh 2t\}$$

Consider $f(t) = \cosh 2t$

$$\mathcal{L}\{\cosh 2t\} = \frac{s}{s^2 - 2^2}$$

Refer Laplace Transform Table for $\cosh \omega t$

$$= \frac{s}{s^2 - 4}$$

$$\therefore \mathcal{L}\{e^{-3t} \cdot \cosh 2t\} = \frac{s}{(s+3)^2 - 4}$$

$e^{-3t}: a = -3$
$\therefore s \rightarrow (s+3)$

Multiplication with t^n

Multiplication with t^n theorem

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$$
$$= (-1)^n \frac{d^n}{ds^n}F(s) \text{ where } n = 1, 2, 3 \dots.$$

Proof of multiplication with t^n

From definition: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Differentiate both side with respect to s

$$\frac{d}{ds}\mathcal{L}\{f(t)\} = \frac{d}{ds}F(s)$$
$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt$$

From Leibniz rule of differentiation under integral sign

$$\frac{dF}{ds} = -\int_0^\infty e^{-st}\,[tf(t)]\, dt$$
$$\frac{dF}{ds} = -\mathcal{L}\{tf(t)\}$$
$$\mathcal{L}\{tf(t)\} = -\frac{dF}{ds} = -F'(s)$$

$$-\int_0^\infty e^{-st}\,[tf(t)]\, dt$$
$$= -\int_0^\infty t\, e^{-st} f(t)\, dt$$
$$= -\mathcal{L}\{tf(t)\}$$

Example 1

By using suitable formula, find $\mathcal{L}\{t\ \cos 3t\}$

Solution :

$$\mathcal{L}\{t\ \cos 3t\}\,, n = 1$$

$$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2 + 3^2}$$

Refer Laplace Transform Table for $\cos \omega t$

$$\mathcal{L}\{t\ \cos 3t\} = (-1)^1 \frac{d}{ds}\left[\frac{s}{s^2 + 3^2}\right]$$

$$= -\left[\frac{(s^2+9)(1) - s(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{s^2 - 2s^2 + 9}{(s^2+9)^2}\right]$$

$$= -\left[\frac{-s^2 + 9}{(s^2+9)^2}\right]$$

$$= \frac{s^2 - 9}{(s^2+9)^2}$$

Differentiate using quotient rule and simplify the equation

Example 2

By using suitable formula, find $\mathcal{L}\{t^2 \ \sin 2t\}$

Solution :

$$\mathcal{L}\{t^2 \ \sin 2t\}\,, n = 2, \omega = 2$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 2^2} = \frac{2}{s^2 + 4}$$

Refer Laplace Transform Table for $\sin \omega t$

$$\mathcal{L}\{t^2 \ \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{2}{s^2 + 4}\right]$$

Differentiate using quotient rule and simplify the equation

$$= \frac{d}{ds}\left[\frac{(s^2 + 4)(0) - 2(2s)}{(s^2 + 4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2 + 4)^2}\right]$$

$$= -\left[\frac{(s^2 + 4)^2(-4) - (-4s)2(s^2 + 4)(2s)}{(s^2 + 4)^4}\right]$$

$$= -(s^2 + 4)\left[\frac{(s^2 + 4)(-4) + 16s^2}{(s^2 + 4)^4}\right]$$

$$= -\left[\frac{(s^2 + 4)(-4) + 16s^2}{(s^2 + 4)^3}\right]$$

$$= \frac{4(4 - 3s^2)}{(s^2 + 4)^3}$$

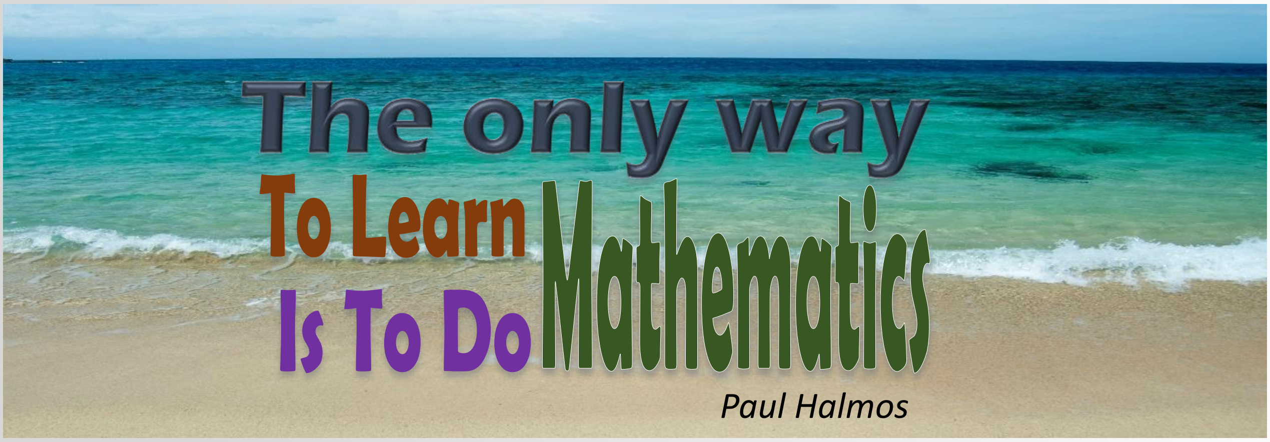
Exercises

By using suitable formula, find the Laplace Transform for the following function

1. $f(t) = -t^3 + 5t - 1$ Answer
2. $f(t) = (t-2)^2$ Answer
3. $f(t) = \dfrac{e^{3t}}{2} - 6t$ Answer
4. $f(t) = -6t - 3e^{-5t}$ Answer
5. $f(t) = 2\sin 3t - \cos 4t$ Answer

Question 6 - 9 (Use first shift theorem)

6. $f(t) = e^{-3t}\ \sin 5t$ Answer
7. $f(t) = e^{-3t}\ \sin 5t + t\sin 4t + 5$ Answer
8. $f(t) = e^{4t}\ \sin\dfrac{2t}{3} - e^{-6t}$ Answer
9. $f(t) = e^{-2t}\cosh 3t$ Answer
10. $f(t) = 2e^{3t}\, sin\, 5t$ Answer
11. $f(t) = t\, cosh\, 2t$ Answer
12. $f(t) = 3t^2\, sin\, t$ Answer



1.3 Apply Inverse Laplace Transform

Inverse Laplace Transform

Inverse Laplace Transform can be defined as the transformation of a Laplace transform that is rational function of s, F(s) into time-domain expression that is function of time, f(t). The inverse Laplace Transform can be written as:

$$\boldsymbol{f(t) = \mathcal{L}^{-1}\{F(s)\}}$$

The inverse can generally be obtained by using standard transforms. By using Laplace Transform Table before, basic properties of the inverse can be used with the standard transforms to obtain a wider range of transforms. Often $F(s)$ is the ratio of two polynomials and cannot be readily identified with a standard transform. However, the use of linearity theorem and shifting theorem can often convert such an expression into simple fraction terms which can then be identified with standard transforms.

Inverse Laplace Transform - Using Laplace Transform Table

From Laplace Transform Table shown before,

$$\mathcal{L}\{a\} = \frac{a}{s}$$

Then inverse Laplace Transform should be:

$$\mathcal{L}^{-1}\left\{\frac{a}{s}\right\} = a$$

$f(t)$	$F(s)$
a	$\frac{a}{s}$

$a \xrightarrow{\text{transform}} \frac{a}{s}$

$a \xleftarrow{\text{inverse}} \frac{a}{s}$

Inverse Laplace Transform - Using Laplace Transform Table

Another example of the reverse process of the inverse Laplace Transform is:

$$\mathcal{L}\{\sin\ \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Then inverse Laplace Transform should be:

$$\mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} = \sin\ \omega t$$

$f(t)$	$F(s)$
$\sin\ \omega t$ $\xrightarrow{\text{transform}}$ $\xleftarrow{\text{inverse}}$	$\frac{\omega}{s^2 + \omega^2}$

Examples and Solution

Determine the inverse Laplace Transform for $F(s) = \frac{3}{4s}$ by using Table of Laplace Transform Table

Solution :

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\left[\frac{3}{4s}\right] = \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right]$$

Since $\frac{a}{s}$ *is equal to a where* $a = 1$ (this question)

Then, $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$$\mathcal{L}^{-1}\left[\frac{3}{4s}\right] = \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right] = \frac{3}{4}$$

$f(t)$	transform → / ← inverse	$F(s)$
a		$\frac{a}{s}$

Examples and Solution

Determine the inverse Laplace Transform for $F(s) = \frac{11}{s^2+4}$ by using Table of Laplace Transform Table

Solution :

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\left[\frac{11}{s^2+4}\right] = 11\mathcal{L}^{-1}\left[\frac{1}{s^2+2^2}\right]$$

Since $sin\ \omega t\ is\ equal\ to\ \frac{\omega}{s^2+\omega^2}\ where\ \omega = 2$

Then, $11\mathcal{L}^{-1}\left[\frac{1}{s^2+2^2}\right] = 11\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\cdot\frac{1}{2}\right]$

$$\mathcal{L}^{-1}\left[\frac{11}{s^2+4}\right] = \frac{11}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] = \frac{11}{2}\ sin\ 2t$$

$f(t)$	$F(s)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

transform

inverse

Examples and Solution

Determine the inverse Laplace Transform for $F(s) = \frac{10}{s^3} - \frac{3}{2s-6}$ by using Table of Laplace Transform Table

Solution :

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\left[\frac{10}{s^3} - \frac{3}{2s-6}\right] = \mathcal{L}^{-1}\left\{\frac{10}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{2s-6}\right\}$$

$\mathcal{L}^{-1}\left\{\frac{10}{s^3}\right\} = 5\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{3}{2s-6}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$

Since $\frac{2}{s^3}$ *is equal to* $\frac{n!}{s^{n+1}}$ *where* $n = 2$ and

$\frac{1}{s-3}$ *is equal to* $\frac{1}{s-a}$ *where* $a = 3$

Then, $5\mathcal{L}^{-1}\left[\frac{2}{s^3}\right] = 5\mathcal{L}^{-1}\left[\frac{2!}{s^{2+1}}\right] = 5t^2$ and

$$\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = \frac{3}{2}e^{3t}$$

$$\mathcal{L}^{-1}\left[\frac{10}{s^3} - \frac{3}{2s-6}\right] = 5t^2 - \frac{3}{2}e^{3t}$$

$f(t)$		$F(s)$
t^n	transform →	$\frac{n!}{s^{n+1}}$
e^{at}	← inverse	$\frac{1}{s-a}$

Examples and Solution

Determine the inverse Laplace Transform for $F(s) = \frac{s+3}{(s-2)^2+16}$ by using Table of Laplace Transform Table

Solution :

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s-2)^2+16}\right] = \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+4^2} + \frac{5}{(s-2)^2+4^2}\right]$$

Then, $\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+4^2}\right] + \mathcal{L}^{-1}\left[\frac{5}{(s-2)^2+4^2}\right]$

For $\omega = 4$ and $\mathrm{a} = -2$, with

e^{at} *cos* ωt *is equal to* $\frac{s-a}{(s-a)^2+\omega^2}$ and

e^{at} *sin* ωt *is equal to* $\frac{\omega}{(s-a)^2+\omega^2}$

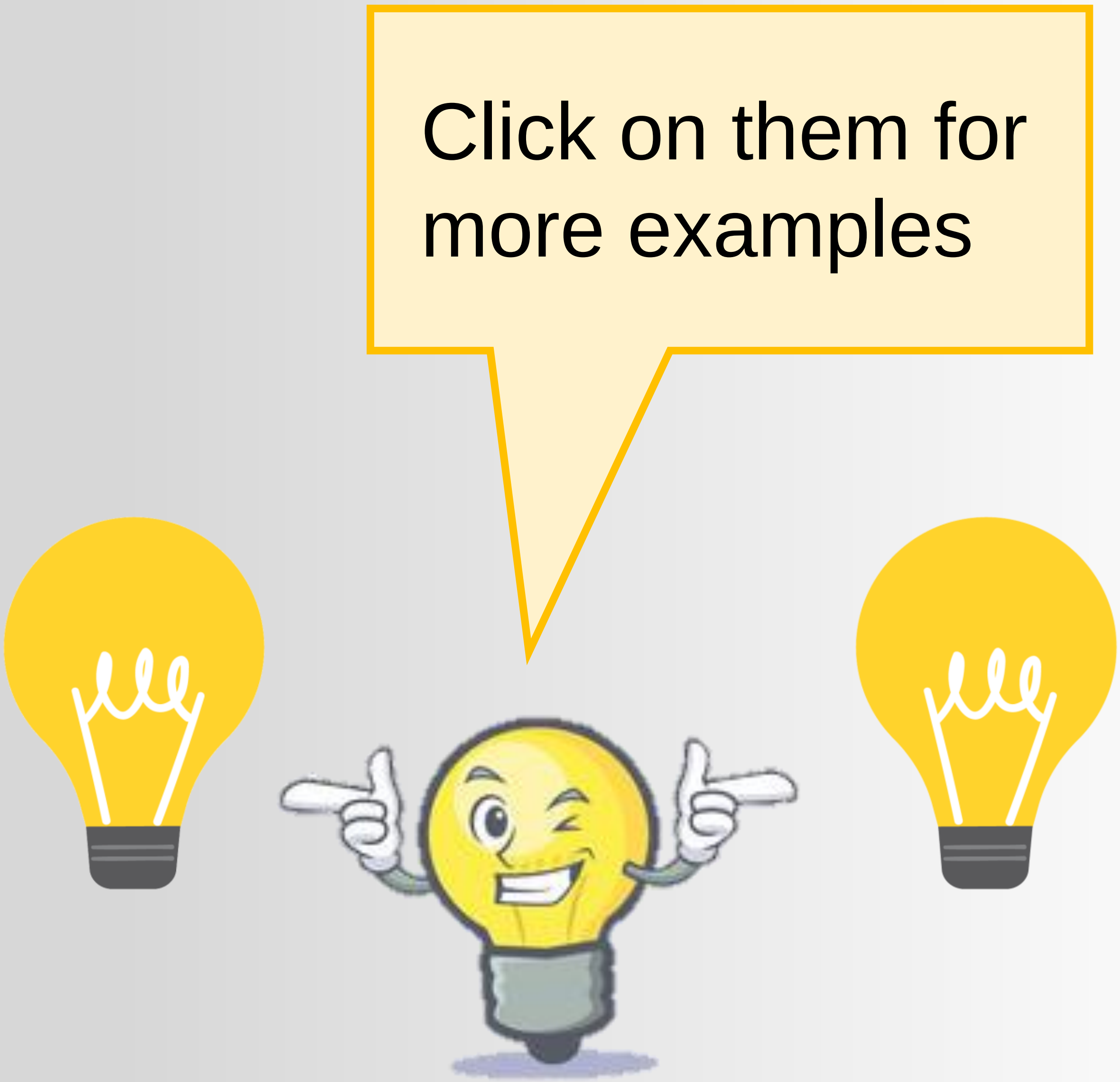
Then, $\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+4^2}\right] = e^{2t}\ \cos 4t$ and

$$\mathcal{L}^{-1}\left[\frac{5}{(s-2)^2+4^2}\right] = 5\mathcal{L}^{-1}\left[\frac{4}{(s-2)^2+4^2} \cdot \frac{1}{4}\right] = \frac{5}{4}e^{2t}\ \sin 4t$$

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s-2)^2+16}\right] = e^{2t}\ \cos 4t + \frac{5}{4}e^{2t}\ \sin 4t$$

$f(t)$	$F(s)$
$e^{-at}\sin\omega t$ transform	$\frac{\omega}{(s+a)^2+\omega^2}$
inverse $e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Examples and Solution



Exercise

Determine the inverse Laplace Transform for $F(s)$ by using Table of Laplace Transform Table. (Click 'answer' to check your answer)

a. $\frac{13}{2s}$ answer

b. $\frac{4}{s+3}$ answer

c. $\frac{3}{2s^5}-\frac{2}{s^3}$ answer

d. $\frac{13}{(s-3)^2}$ answer

e. $\frac{2s^4+4}{s^6}$ answer

f. $\frac{10}{s^2-16}$ answer

g. $\frac{s+6}{(s+2)^2+16}$ answer

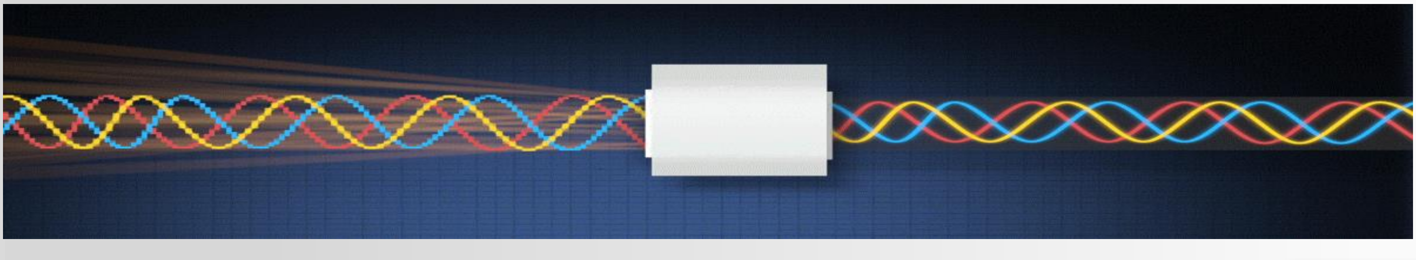
h. $\frac{8s}{3s^2-4}$ answer

i. $\frac{1}{s^2-5s+6}$ answer

j. $\frac{3s+5}{16s^2-9}$ answer

k. $\frac{1}{(s-3)^5}$ answer

l. $\frac{8}{s^2+64}+\frac{3s}{s^2+64}-\frac{1}{(s-2)^2}$ answer



Inverse Laplace Transform – Using Partial Fraction

In other way, if we need to find the inverse transform of a function that has the form:

$$F(s)=\frac{N(s)}{D(s)}=\frac{a_n s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\ldots+as+1}{b_m s^m+b_{m-1}s^{m-1}+b_{m-2}s^{m-2}+\ldots+bs+1}$$

The best solution is by using partial fraction. By using this technique a complicated fraction was split up into forms that are in the Laplace Transform table.

Previous Lessons:

How to construct partial fraction?

- ✓ The degree of the numerator must be less than the denominator. Otherwise we have to long division.
- ✓ Write denominator in prime factor. This will be the patern of partial fraction.

What kind of partial fraction?

- ✓ Linear factor
- ✓ Repeated Linear Factor
- ✓ Quadratic Factor
- ✓ Repeated Quadratic Factor

Examples and Solution

Determine the inverse Laplace Transform for $F(s) = \frac{x+5}{x^3-x}$

Solution :

$$f(t) = \mathcal{L}^{-1}\left\{\frac{x+5}{x^3-x}\right\}$$

$$\frac{x+5}{x^3-x} = \frac{x+5}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \quad \text{(linear factor)}$$

$$x + 5 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$x = 0,$	$x = 1,$	$x = -1,$
$5 = A(-1)(1)$	$1 + 5 = B(1)(1+1)$	$-1 + 5 = C(-1)(-1-1)$
$A = -5$	$B = 3$	$C = 2$

$$\frac{x+5}{x(x-1)(x+1)} = -\frac{5}{x} + \frac{3}{x-1} + \frac{2}{x+1}$$

$$\mathcal{L}^{-1}\left\{-\frac{5}{x} + \frac{3}{x-1} + \frac{2}{x+1}\right\} = \mathcal{L}^{-1}\left\{-\frac{5}{x}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{x-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{x+1}\right\}$$

$$\mathcal{L}^{-1}\left\{-\frac{5}{x}\right\} = -5 \quad \oplus \quad \mathcal{L}^{-1}\left\{\frac{3}{x-1}\right\} = 3e^{t} \quad \oplus \quad \mathcal{L}^{-1}\left\{\frac{2}{x+1}\right\} = 2e^{-t}$$

Then, $\mathcal{L}^{-1}\left\{\frac{x+5}{x^3-x}\right\} = -5 + 3e^{t} + 2e^{-t}$

Examples and Solution

Determine the inverse Laplace Transform for $F(s) = \frac{s^2-2}{(s-2)(s+1)^3}$

Solution :

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s^2-2}{(s-2)(s+1)^3}\right\}$$

$$\frac{s^2-2}{(s-2)(s+1)^3} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$ (repeated linear factor)

$$s^2 - 2 = A(s+1)^3 + B(s-2)(s+1)^2 + C(s-2)(s+1) + D(s-2)$$

$$s^2 - 2 = (s^3 + 3s^2 + 3s + 1) + B(s^3 - 3s - 2) + C(s^2 - s - 1) + D(s - 2)$$

$s = 2,$	$s = -1,$	$Compare\ coeficient\ of\ s^3,$	$Compare\ coeficient\ of\ s^2,$
$A = \frac{2}{27}$	$D = \frac{1}{3}$	$B = -\frac{2}{27}$	$C = \frac{7}{9}$

$$\frac{s^2-2}{(s-2)(s+1)^3} = \frac{2}{27(s-2)} - \frac{2}{27(s+1)} + \frac{7}{9(s+1)^2} + \frac{1}{3(s+1)^3}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2-2}{(s-2)(s+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{27(s-2)} - \frac{2}{27(s+1)} + \frac{7}{9(s+1)^2} + \frac{1}{3(s+1)^3}\right\}$$

$$= \frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} - \frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} + \frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$\frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} = \frac{2}{27}e^{2t}, \quad -\frac{2}{27}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} = -\frac{2}{27}e^{-t}, \quad \frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \frac{7}{9}te^{-t}$$

$$and \quad \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} = \frac{1}{3}\left[\frac{n!}{\left(s-(-a)\right)^{n+1}}\right] \quad where\ \ n = 2\ \ and\ a = -1$$

$$\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} = \frac{1}{3}\left[\frac{2!}{(s+1)^{2+1}}\right] = \frac{1}{3}t^2e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2-2}{(s-2)(s+1)^3}\right\} = \frac{2}{27}e^{2t} - \frac{2}{27}e^{-t} + \frac{7}{9}te^{-t} + \frac{1}{3}t^2e^{-t}$$

Examples and Solution

Determine the inverse Laplace Transform for $F(s) = \frac{s^3-2}{s^4-1}$

Solution :

$f(t) = \mathcal{L}^{-1}\left\{\frac{s^3-2}{(s^2+1)(s+1)(s-1)}\right\}$ hint: $s^4 - 1 = (s^2+1)(s^2-1)$ and $s^2 - 1 = (s+1)(s-1)$

$\frac{s^3-2}{(s^2+1)(s+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{(s-1)}$ (quadratic factor)

$$s^3 - 2 = (As+B)(s-1)(s+1) + C(s-1)(s^2+1) + D(s^2+1)(s+1)$$

$$s^3 - 2 = As^3 - As + Bs^2 - B + C(s^3 - s^2 + s - 1) + D(s^3 + s^2 + s + 1)$$

$s = -1,$ $s = 1,$ $Compare\ coeficient\ of\ s^3,$ $Compare\ coeficient\ of\ s^2$,

$C = \frac{3}{4}$ $D = -\frac{1}{4}$ $A = \frac{1}{2}$ $B = 1$

$$\frac{s^3-2}{(s^2+1)(s+1)(s-1)} = \frac{\frac{1}{2}s+1}{s^2+1} + \frac{\frac{3}{4}}{s+1} - \frac{\frac{1}{4}}{(s-1)} = \frac{1}{4}\left[\frac{2s+4}{s^2+1} + \frac{3}{s+1} - \frac{1}{(s-1)}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{s^3-2}{s^4-1}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+1} + \frac{3}{s+1} - \frac{1}{(s-1)}\right\}$$

$\mathcal{L}^{-1}\left\{\frac{s^3-2}{s^4-1}\right\} = \frac{1}{4}\left[\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\}\right]$ where

$$\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 2cos\ \omega t + 4sin\ \omega t$$

$3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 3e^{-t}$ and $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} = e^{t}$

$$\mathcal{L}^{-1}\left\{\frac{s^3-2}{s^4-1}\right\} = \frac{1}{4}[2cos\ \omega t + 4sin\ \omega t + 3e^{-t} - e^{t}]$$

Examples and Solution

Determine the inverse Laplace Transform for $F(s)$. (Click 'answer' to check your answer)

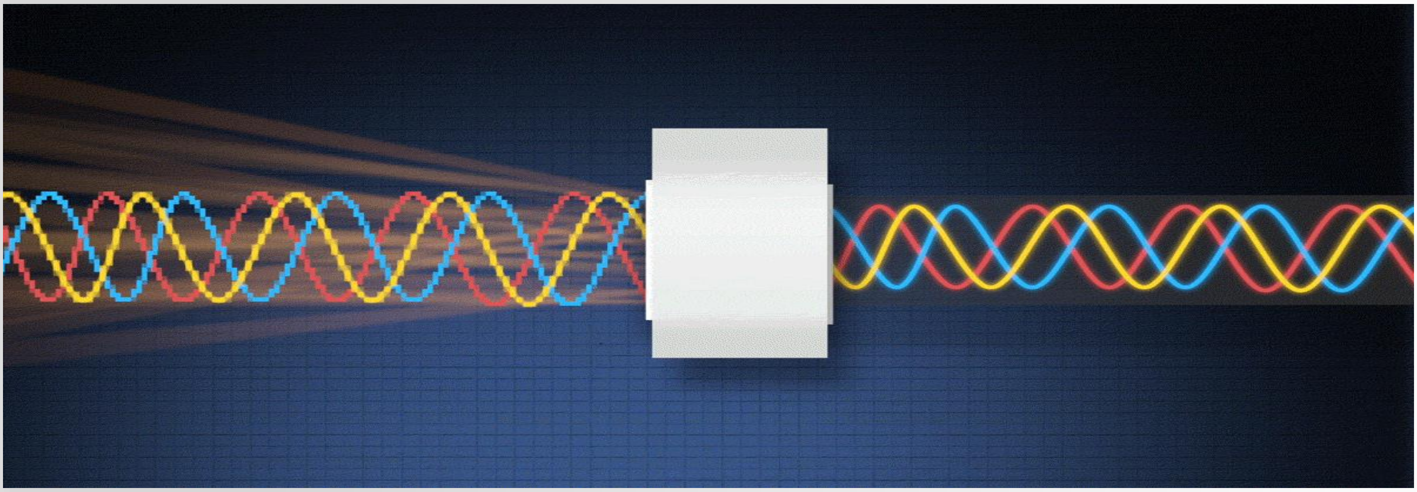
a. $\dfrac{s^2+10}{s(s+2)(s-1)}$ answer

b. $\dfrac{s^2}{(s+1)(s^2+1)}$ answer

c. $\dfrac{3}{4(s^2-9)}$ answer

d. $\dfrac{s-3}{2s^2(s-1)^2}$ answer

e. $\dfrac{s+4}{s^2-3s+2}$ answer



Inverse Laplace Transform – Laplace Transform of Derivatives

At previous section we have seen how to find the Laplace Transform and its inverse. This techniques are now applied to finding the solution to differential equations. By applying the transform, the differential equations is converted into an algebraic equation. This algebraic equation is solved and then the inverse Laplace Transform is applied to solve the differential equation. The advantage of using the Laplace transform is that initial conditions are automatically incorporated into the solution.

Let $f(t)$ be a function of f and let $F(s)$ be the Laplace Transform of f. The value of f and its derivatives when $t = 0$ are denoted by $f(0), f'(0), f"(0)$ and so on. The n th derivative of f is denoted by $f^{(n)}(t)$. Then it can be shown that the Laplace Transform of $f^{(n)}(t)$ is given by:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'^{(0)} - \ldots\ldots - f^{(n-1)}(0)$$

**in this case we only discuss for $n = 1$ and $n = 2$*

Inverse Laplace Transform – Laplace Transform of Derivatives

From the previous lesson, we know that:

$$\mathcal{L}\{f(t)\} = F(s)$$

First order derivatives:

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ or refer to formula:

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
$\frac{dy}{dt}, y'(t)$	$sY(s) - y(0)$

Second order derivatives:

$\mathcal{L}\{f'(t)\} = s^2F(s) - sf(0) - f'(0)$ or refer to formula:

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
$\frac{d^2y}{dt^2}, y''(t)$	$s^2Y(s) - sy(0) - y'(0)$

Examples and Solution

Write the expression of Laplace Transform of $f'(t)$ given that $f(0) = 2$

Solution :

By using formula: $y'(t) = sY(s) - y(0)$

Then, $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ where $f(0) = 2$

$\mathcal{L}\{f'(t)\} = sF(s) - 2$

Click on me for more explanation on how to do Laplace Transform of derivatives

Examples and Solution

Write the expression of Laplace Transform of $\boldsymbol{3f''(t)+2f'(t)-0.5f(t)}$ given that $\boldsymbol{f(0)=2}$ and $\boldsymbol{f'(0)=-1}$

Solution :

By using formula: $\mathcal{L}\{f(t)\}=F(s)$,

$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$ and

$\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$

Then,

$\mathcal{L}\{3f''(t)\}=3[s^2F(s)-sf(0)-f'(0)]$ where $f(0)=2$ and $f'(0)=-1$

$=3[s^2F(s)-s(2)-(-1)]$

$=3[s^2F(s)-2s+1]$

$\mathcal{L}\{2f'(t)\}=2[sF(s)-f(0)]$ where $f(0)=2$

$=2[sF(s)-2]$

$\mathcal{L}\{0.5f(t)\}=0.5F(s)$

Finally $\mathcal{L}\{3f''(t)+2f'(t)-0.5f(t\}$

$=3[s^2F(s)-2s+1]+2[sF(s)-2]-0.5F(s)$

$=3s^2F(s)-6s+3+2sF(s)-4\ -0.5F(s)$

$=3s^2\ F(s)+2sF(s)-0.5F(s)-6s-1$ or

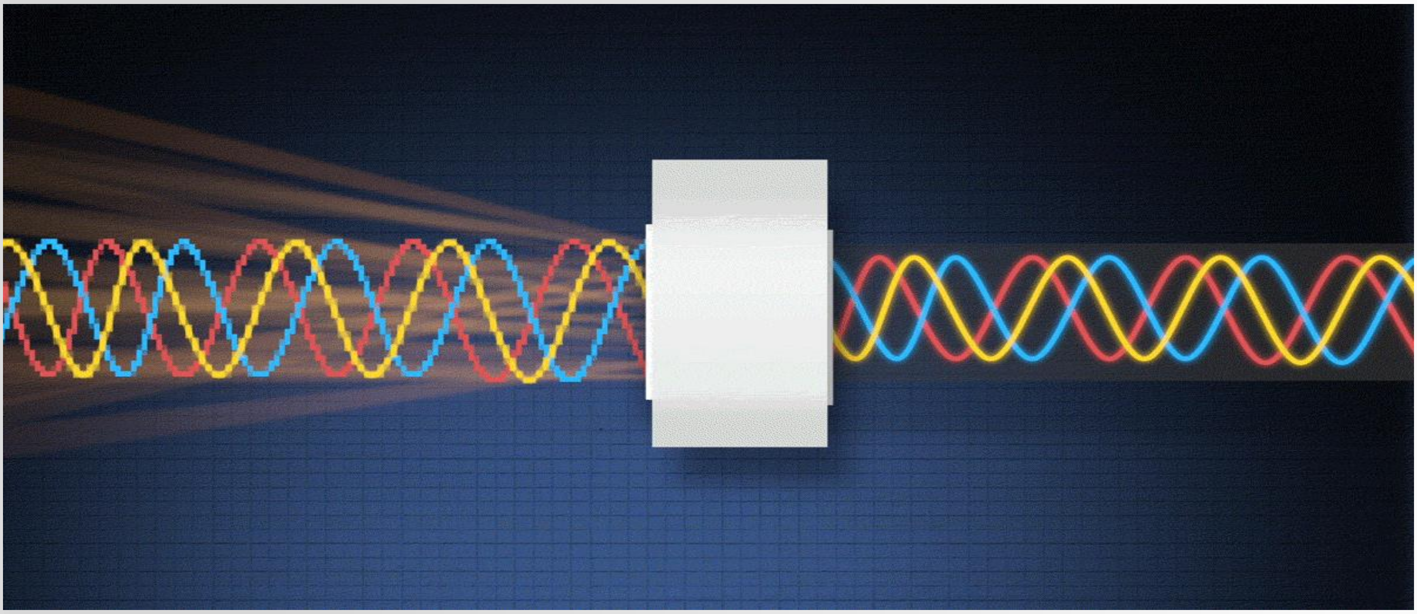
$=[\ 3s^2+2s-0.5]F(s)-6s-1$

Excercise

The Laplace Transform of $y(t)$ is $Y(s), y(0) = 2$ and $y'(0) = -1$. Obtain an expression for the Laplace transform of the following functions.
(Click 'answer' to review the answer)

a. y'' answer

b. $y' + \frac{y}{2}$ answer



Solving Differential Equations

Solve $y' - y = 3 \; with \; y(0) = 2$

Solution :

By using formula: $\mathcal{L}\{y(t)\} = Y(s)$ and

$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

Then,

$sY(s) - y(0) - Y(s) = \mathcal{L}\{3\}$

$Y(s)[s-1] - 2 = \frac{3}{s}$

$Y(s)[s-1] = \frac{3}{s} + 2$

$Y(s) = \frac{3+2s}{s} \cdot \frac{1}{s-1}$

$Y(s) = \frac{3+2s}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$ (use partial fraction)

$3 + 2s = A(s+1) + Bs$

$s = 0,$ $\quad$ $s = 1$
$A = -3$ $\quad$ $B = 5$

$3 + 2s = A(s+1) + Bs$

$Y(s) = \frac{-3}{s} + \frac{5}{s-1}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{-3}{s} + \frac{5}{s-1}\right\}$

$\mathcal{L}^{-1}\left\{\frac{-3}{s} + \frac{5}{s-1}\right\} = -3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$\therefore y(t) = -3 + 5e^{t}$

Solving Differential Equations

Solve $y'' + y' - 2y = 0$ *with* $y(0) = 2$ *and* $y'(0) = -1$

Solution :

By using formula: $\mathcal{L}\{y(t)\} = Y(s)$ and

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

Then,

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) = 0$$

$$s^2Y(s) - s(2) - (-1) + sY(s) - (2) - 2Y(s) = 0$$

$$Y(s)[s^2 + s - 2] - 2s - 2 + 1 = 0$$

$$Y(s)[s^2 + s - 2] = 2s + 1$$

$$Y(s) = \frac{2s+1}{s^2+s-2} = \frac{2s+1}{(s-1)(s+2)}$$

$$\frac{2s+1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$ (use partial fraction)

$$2s + 1 = A(s + 2) + B(s - 1)$$

$s = 1,$ $\quad s = -2$

$A = 1$ $\quad B = 1$

$$Y(s) = \frac{1}{s-1} + \frac{1}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{s+2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\therefore y(t) = e^t + e^{-2t}$$

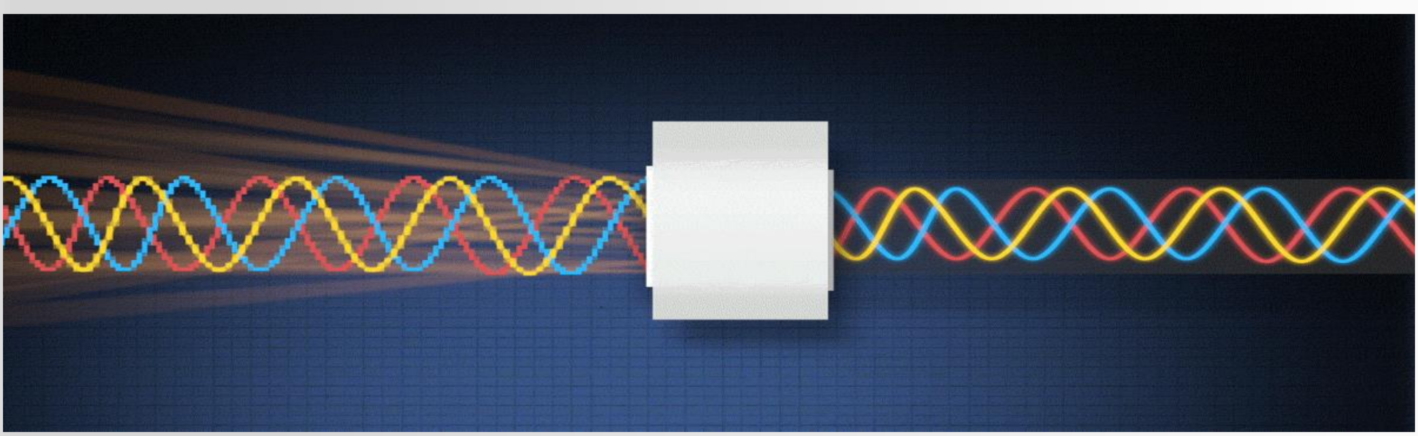
Excercise

The Laplace Transform of $y(t)$ is $Y(s), y(0) = 2$ and $y'(0) = -1$.
Solve the following equations by using Laplace Transform.
(Click 'answer' to review the answer)

a. $y' + 2y = sin\ 3t$ answer

b. $y'' - 2y' + y = e^{-t}$ answer

Solve the differential equation
$y' + 3y = e^{-4t}$ given that $y(0) = 0$
(Click on me to see the way I do)

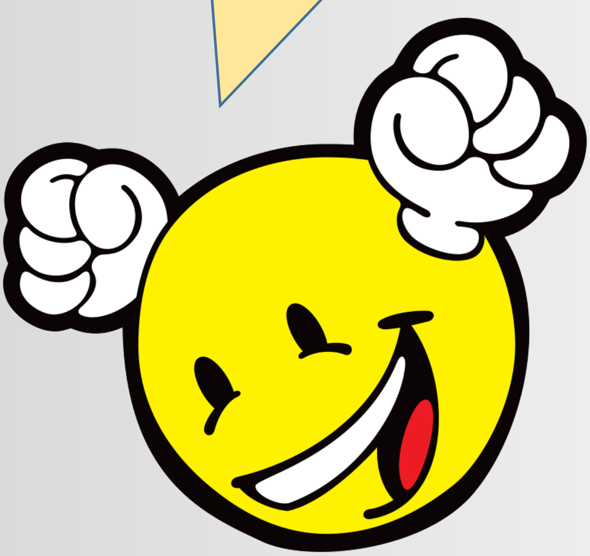


Self Test

Let’s fun with Laplace Transform

Click on me when you are ready.

- Double your score by clicking **‘x2 score’** button
- Click **‘extra time’** button if you an need extra time



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